AWCBB

- AWCBB (Automata with constraints Between Brothers):
  - AWE DC in which constraints are restricted to \( i = j, i \neq j \) (\( i, j \in N \))

- Ex.: \( A = ((q, F, \{q\}, \Delta) \)
  - \( F = \{f(., .)\} \)
  - \( \Delta = \{a \rightarrow q, f(q, q) \mapsto q\} \)
  - \( L(A) \) is the set of complete binary trees

- AWCBB is closed by union, intersection, and complement

- Emptiness problem for AWCBB is decidable

- Key of the proof:
  - Consider a deterministic AWCBB. Then if \( q_1 \neq q_2 \) the following rule never be used
    \[ f(q_1, q_2) \mapsto q \]
  - For the following rule, we must know whether more than one trees are reachable to the given state
    \[ a \rightarrow q, b \rightarrow q, f(q, q) \mapsto q' \]

- \( M_F \): maximum number of arguments of symbols in \( F \)
- \( L(q) \) \( \overset{\text{def}}{=} \{ t \mid t \rightarrow_A q \} \)

- Th.: Emptiness for an AWCBB is decidable

- Proof sketch: Assume deterministic AWCBB with rule set \( \Delta \)
  - Initialize \( L_0 := \emptyset \), and repeat the following step for each state \( p \) until \( L_p \)'s saturate
  - \( L_q := L_q \cup \{ t \} \) for \( t \) such that \( t \notin L_q \), \( |L_q| \leq M_F \) and the following holds:
    \[ f(q_1, \ldots, q_n) \mapsto q \in \Delta, \quad t_1 \in L_{q_1}, \ldots, t_n \in L_{q_n}, \quad \text{and} \]
    \[ t = f(t_1, \ldots, t_n), \quad t \models c \]
  - It is empty if \( L_q = \emptyset \) for all accepting states \( q \)

Reduction automata

- AWEDC that satisfies the conditions
  - States are ordered <, and
  - For any \( f(q_1, \ldots, q_n) \mapsto q \)
    \[ \forall i. q_i > q \text{ if } c \text{ contains equality} \]
    \[ \forall i. q_i \geq q \text{ otherwise} \]

- Ex.: Reduction automaton accepting \( g(g(t, s), t) \) for any \( t, s \)
  - \( a \rightarrow q_t, g(q_t, q_t) \mapsto qy_0, g(q_t, q_t) \mapsto qy_0 \)
  - \( g(qy_0, qy_0) \mapsto qf, g(qy_0, qy_0) \mapsto qy_0 \)
  - \( g(qy_0) \mapsto qf, g(qy_0, qy_0) \mapsto qy_0 \)
  - \( g(qy_0) \mapsto qf, g(qy_0, qy_0) \mapsto qf \) where \( q \in \{ q_t, q_y, q_f \} \)

Property of reduction automata

- Closed under union and intersection
  - Open for complement

- Emptiness problem
  - Decidable if complete and deterministic
  - Undecidable if non-deterministic

- Finiteness problem (finiteness of \( L(A) \)) is decidable